1 Classical Probability

1.1 Theoretical Experiment

- In a **'theoretical experiment'** we consider running an experiment like flipping coin a fair coin 3 times.
- **Sample Space** - The set of all possible outcomes of an experiment. For this section, it is important to write all outcomes as 'equally likely'.
- **Event** - A subset of the Sample Space which represents all outcomes which satisfy a condition.
- **Probability** - A function like $f(x)$ which maps an event to the ratio of the number of outcomes in the event to the number of outcomes in the Sample Space. That is:

  $$P(\text{Event}) = \frac{n(\text{Event})}{n(\text{Sample Space})} = \frac{\text{Number of favorable outcomes}}{\text{Number of possible outcomes}}$$

  Note: *Since the set of favorable outcomes is at least empty and is at most the Sample Space, all probabilities are between* 0 *and* 1 *inclusive.*
- **Enumeration** - When the Sample Space is not large, it is helpful to write out the entire set.

1.1.1 Example

You flip a fair coin three times. Find the following probabilities.

(a) What is the probability of flipping at least 2 heads.

(b) What is the probability of flipping three heads OR three tails.

– First let's write out the Sample Space. We want to make sure that each outcome is equally likely. For instance (2 heads - 1 tail) occurs 3 different ways, whereas (3 heads - 0 tail) only occurs 1 way. So we want to write the Sample Space accounting for all the possible ways the outcomes can occur:

$$S = \{hhh, hht, hth, thh, htt, tht, tth, ttt\}$$

(a) The event "At least 2 heads" is the set $\{hhh, hht, hth, thh\}$.
Thus the probability is:

$$P(\text{At least 2 heads}) = \frac{n(\text{At least 2 heads})}{n(\text{Sample Space})}$$

$$= \frac{n(\{hhh, hht, hth, thh\})}{n(\{hhh, hht, hth, thh, htt, tht, tth, ttt\})} = \frac{4}{8} = \frac{1}{2}$$

(b) The event "Three heads OR three tails" is: $\{hhh, ttt\}$

Thus: $P(\text{Three heads or three tails}) = \frac{2}{8} = \frac{1}{4}$

1.2 Empirical Probability

- **Empirical Probability** - Instead of running a 'theoretical experiment', we collect real data. Here the number of favorable outcomes becomes simply the frequency of data values which meet the condition. The Sample Space becomes the sample of data points we collected.
- **Venn Diagram** - A diagram which shows the **number** of elements in particular events (or probabilities those events occur) of the Sample Space. The events are represented by circles and the Sample Space is represented by a rectangle.
- **A AND B** - An event which meets BOTH conditions A and B.
- **A OR B** - An event which meets at least one of the two conditions. For instance Algebra OR Biology consists of all students taking Algebra but not Biology, Biology but not Algebra, or BOTH Algebra and Biology.
- **Conditional Probability** - The probability of A GIVEN B, written as $P(A|B)$ is probability event A occurs when you shrink the Sample Space to the given event B.

1.2.1 Example

There are 140 freshmen in Anytown High School. The students can enroll in Algebra or Biology. (Note, it is possible to enroll in BOTH Algebra and Biology, as well not enroll in either Algebra nor Biology). 80 students take Algebra, 30 students take Biology, 10 students take BOTH Algebra and Biology.

(a) What is the probability a randomly selected student takes Algebra?

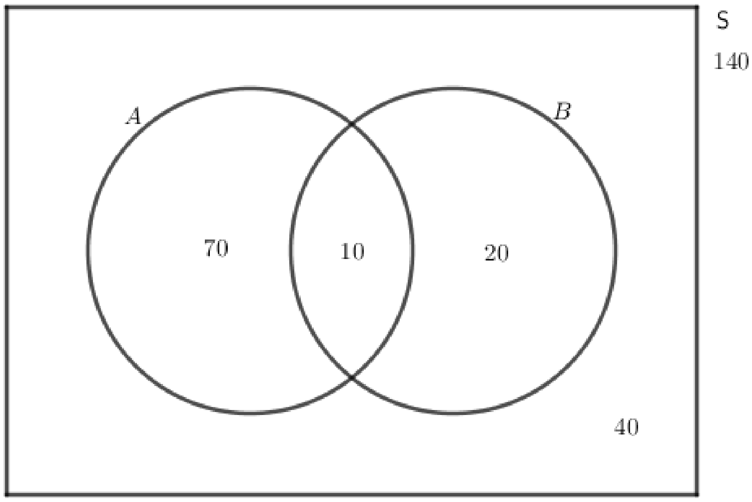
(b) What is the probability a randomly selected student takes Algebra AND Biology?

(c) What is the probability a randomly selected student takes Algebra OR Biology?

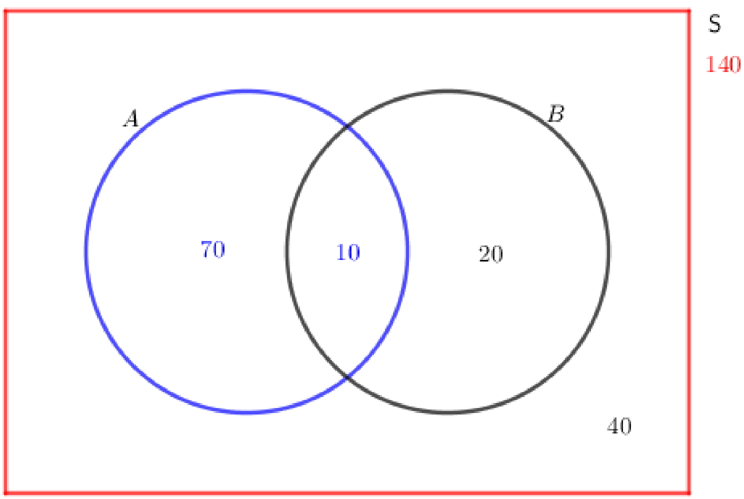
(d) What is the probability a randomly selected student takes Algebra GIVEN they take Biology?

(e) What is the probability a randomly selected Algebra student also takes Biology?

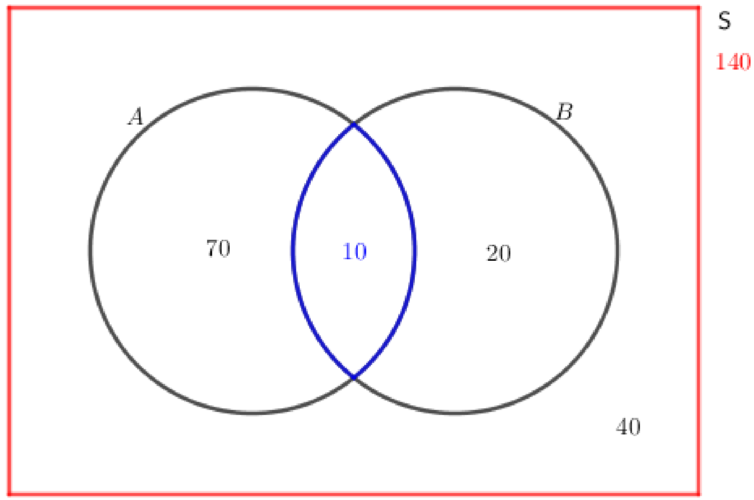
– First let us populate a Venn Diagram. It is best to start at the intersection and work out from there.



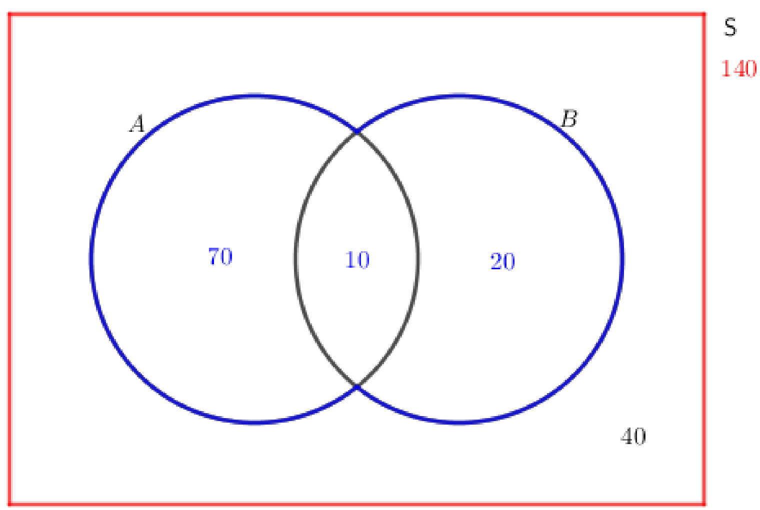
(a) $P(A) = \frac{n(A)}{n(S)} = \frac{80}{140} = \frac{4}{7}$



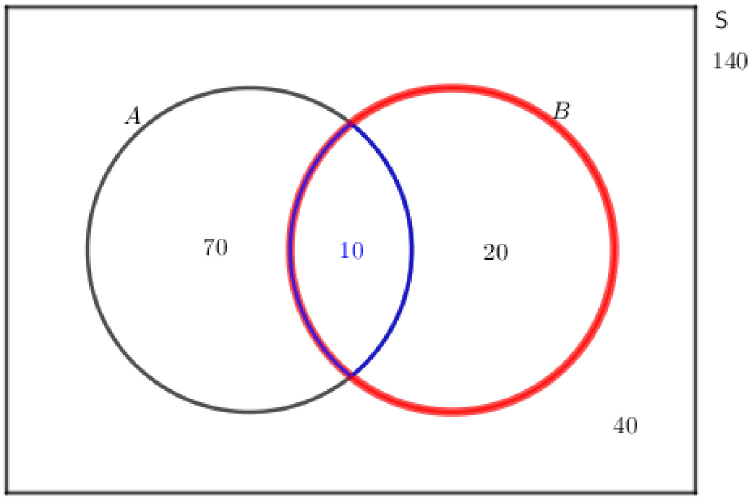
(b) $P(A\ AND\ B) = \dfrac{n(A\ AND\ B)}{n(S)} = \dfrac{10}{140} = \dfrac{1}{14}$



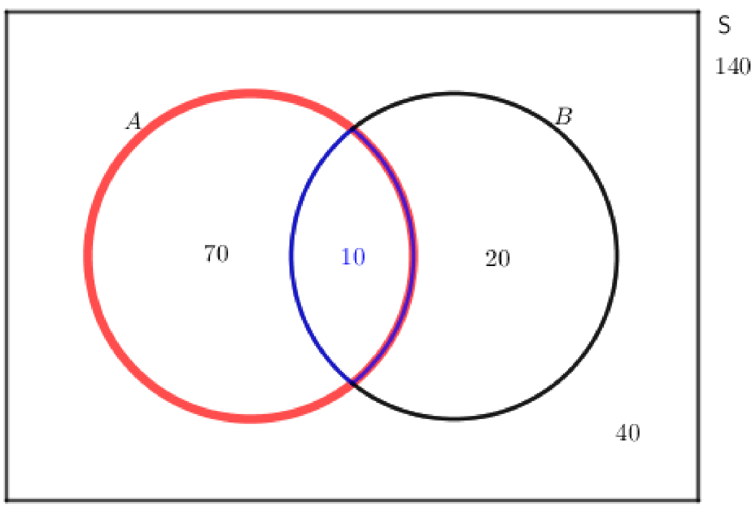
(c) $P(A\ OR\ B) = \dfrac{n(A\ OR\ B)}{n(S)} = \dfrac{100}{140} = \dfrac{5}{7}$



(d) Here we are 'shrinking' the Sample Space to what is given, in this case B. $P(A|B) = \frac{n(A\ AND\ B)}{n(B)} = \frac{10}{30} = \frac{1}{3}$



(e) Although we are not using the word 'GIVEN', this is still conditional on the student taking Algebra. $P(B|A) = \frac{n(B\ AND\ A)}{n(A)} = \frac{10}{80} = \frac{1}{8}$



- **Contingency Table** - Basically, it is just a Venn Diagram in a table format. That is, instead of circles for the events, they are rows and columns. Often, the row and column totals are written on the outside of the table.

1.2.2 Example

The following table displays the data for the sex and highest degree earned for the participants of a scientific conference.

	B.S.	M.S.	Ph.D.	
M	**10**	**15**	**25**	50
F	**10**	**13**	**12**	35
	20	28	37	85

(a) Probability a randomly selected participant's highest degree earned is an M.S. degree?

– $P(M.S.) = \dfrac{n(M.S.)}{n(S)} = \dfrac{28}{85}$

(b) Probability a randomly selected participant's highest degree earned is an M.S. degree OR they're female?

– $P(M.S.\ OR\ F) = \dfrac{n(M.S.\ OR\ F)}{n(S)} = \dfrac{50}{85} = \dfrac{10}{17}$

(c) Probability a randomly selected participant's highest degree earned is an M.S. degree AND they're female?

– $P(M.S.\ AND\ F) = \dfrac{n(M.S.\ AND\ F)}{n(S)} = \dfrac{13}{85}$

(d) Probability a randomly selected female's highest degree earned is an M.S. degree?

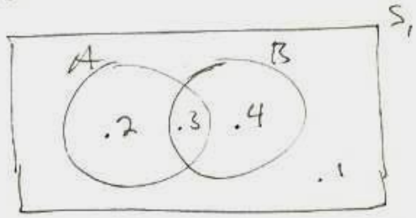
– $P(M.S.|F) = \dfrac{n(M.S.\ AND\ F)}{n(F)} = \dfrac{13}{35}$

1.3 Formulas, Independence, and Mutual Exclusivity

- Complement Rule: $P(NOT\ A) = 1 - P(A)$
- Addition Rule: $P(A\ OR\ B) = P(A) + P(B) - P(A\ AND\ B)$
- Multiplication Rule: $P(A\ AND\ B) = P(A) \cdot P(B|A)$
- A, B are called **Mutually Exclusive** iff $A\ AND\ B$ is empty. This means that the two events cannot happen at the same time. It also implies that $P(A\ AND\ B) = 0$. Example: If A is "I eat pizza for lunch", and B is "I eat a burger for lunch", A, B are mutually exclusive because I cannot eat BOTH a burger and pizza for lunch. That is, if I had pizza, I didn't have a burger. If I had a burger, I didn't have pizza. I could have had neither. But I could NOT have had BOTH.
- A, B are called **independent** iff $P(B|A) = P(B)$. Equivalently, $P(A|B) = P(A)$. Example: Let A be the event "It rains in China" and B be the event "I eat pizza for lunch". $P(B|A) = P(B)$ because whether or not it rains in China does NOT affect my lunch. A and B can happen at the same time, what is important here is the probability I eat pizza for lunch DOES NOT CHANGE when given information that it rains in China.

1.3.1 Three Venn Diagrams

General



$p(A) = .5$

$p(B) = .7$

$p(B|A) = .3/.5 = .6$

$p(A \text{ and } B) = .3$

$p(A \text{ or } B) = .9$

M.R.

$p(A \text{ and } B) = p(A)\, p(B|A)$

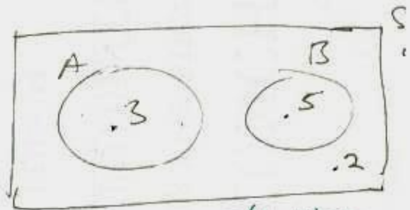
$.3 = (.5)(.6)$

A.R.

$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$

$.9 = .5 + .7 - .3$

Mutually Exclusive



$p(A \text{ and } B) = 0$

$p(A) = .3$

$p(B) = .5$

$p(B|A) = 0$

$p(A \text{ and } B) = 0$

$p(A \text{ or } B) = .8$

M.R.

$p(A \text{ and } B) = 0$

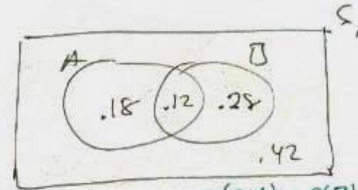
$0 = 0$

A.R.

$p(A \text{ or } B) = p(A) + p(B)$

$.8 = .3 + .5$

Independent



$p(B|A) = p(B)$

$p(A) = .3$

$p(B) = .4$

$p(B|A) = \frac{.12}{.3} = .4$

$p(A \text{ and } B) = .12$

$p(A \text{ or } B) = .58$

M.R.

$p(A \text{ and } B) = p(A)\, p(B)$

$.12 = (.3)(.4)$

A.R.

$p(A \text{ or } B) = p(A) + p(B) - p(A)p(B)$

$.58 = .3 + .4 - .12$

1.4 Sequential Trials

- **Tree Diagram** - a diagram which starts at a point of origin and connects to sequential events in series.
  - The **conditional probabilities** are written on the arcs.
  - The sum of the probabilities on all arcs directly out of a node will always add to 1
  - Use the multiplication rule to find the probability of ending at a particular terminal node.
- Sample marbles from a jar **without replacement** - Build a tree diagram. Note that the marbles are 'used up' so the conditional probabilities change at every tier along the tree diagram
- Sample marbles from a jar **with replacement** - Build a tree diagram. Note that the marbles are replaced each time, so the conditional probabilities are the same at every tier along the tree diagram.

1.4.1 Example

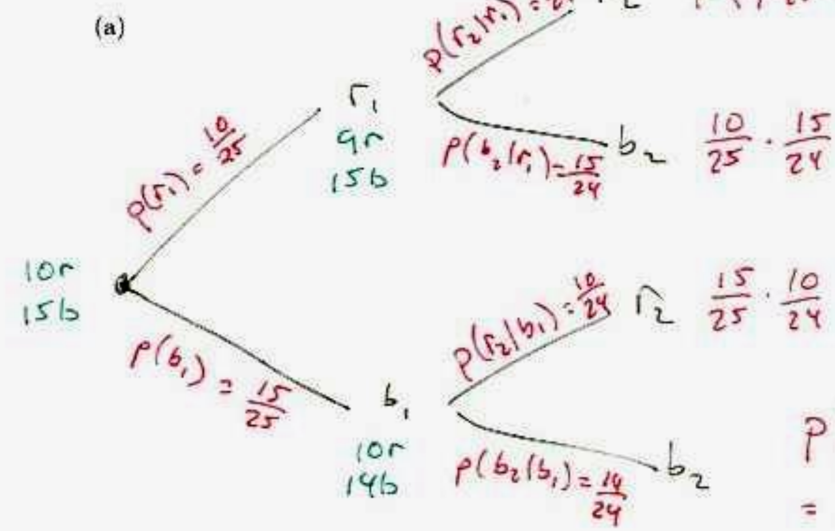
A jar contains 10 red and 15 blue marbles.

(a) You draw 2 marbles without replacement. What is the probability at least 1 is red?

(b) You draw 2 marbles with replacement. What is the probability at least 1 is red?

(a)

$P(r_1 r_2) = P(r_1)P(r_2|r_1) = \frac{10}{25} \cdot \frac{9}{24}$

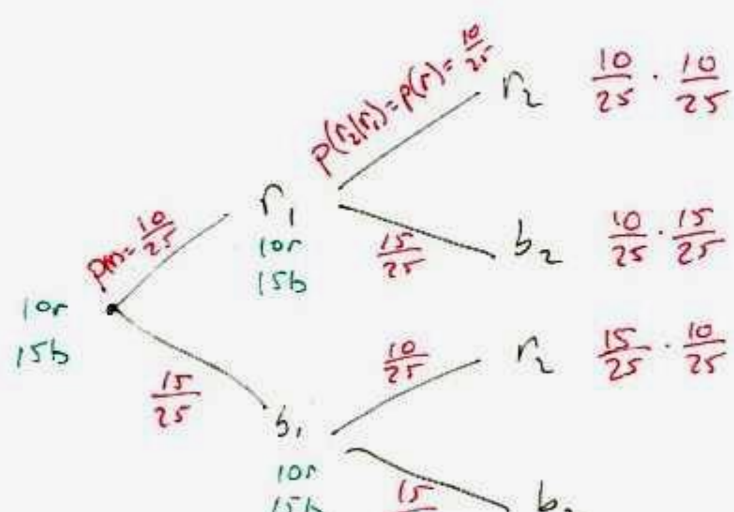


$P(\text{at least 1 red})$

$= \frac{10}{25} \cdot \frac{9}{24} + \frac{10}{25} \cdot \frac{15}{24} + \frac{15}{25} \cdot \frac{10}{24}$

$= .65$

(b)



$P(\text{at least 1 red})$

$= \frac{10}{25} \cdot \frac{10}{25} + \frac{10}{25} \cdot \frac{15}{25} + \frac{15}{25} \cdot \frac{10}{25} = .64$

1.5 Counting

- Fundamental Counting Principles
  - If sets are disjoint (meaning they don't overlap)
    OR means +
  - If choices are independent (meaning your previous decision doesn't affect the number of choices you have now)
    AND means $\times$
- **Factorial,** $n!$ - Number of ways to arrange n items in a row.

$$n! = n(n-1)(n-2)...(1) \qquad\qquad 0! = 1$$

- **Permutation,** nPr - Number of ways to arrange r items in a row from a pool of n. ORDER MATTERS HERE!

$$nPr = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

  Note, this is a factorial CUT SHORT with only r factors.

- **Combination,** nCr - Number of ways to CHOOSE r items from a pool of n. ORDER DOES NOT MATTER HERE!

$$nCr = \frac{nPr}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

  Note: $nC0 = 1, nC1 = n, nCn = 1, nCr = nC(n-r)$

1.5.1 Example

1. There are 5 entrees, 6 desserts, and 4 coffees. How many ways can you select: (an entree) AND [(a dessert) OR (a coffee)]?

– $(5) \times [(6) + (4)] = 50$

2. How many ways can you put 4 items on a shelf?

– $4! = 4 * 3 * 2 * 1 = 24$

3. How many ways can you arrange 4 decorations on a shelf from a box a 10?

– $10P4 = 10 * 9 * 8 * 7 = 5040$

4. How many ways can you select a President, Vice President, Secretary and Treasurer from a class of 10?

– ORDER MATTERS, so this is the same as above. $10P4 = 10 * 9 * 8 * 7 = 5040$

5. How many ways can you select 4 senators from a class of 10?

– ORDER DOES NOT MATTER, so this is $10C4 = 5040/24 = 210$

6. There are 5 women and 3 men. 3 are chosen at random. What is the probability 2 women and 1 man were selected?

– $P(2w1m) = \dfrac{\text{\# ways to select 2 women from a pool of 5 women AND 1 man from a pool}}{\text{\# ways to select 3 people}}$

$= \dfrac{(5C2)(3C1)}{8C3} \approx .5357$