Problem 42

Let $f(x) = -x(x-1)(x-2)(x-a)(x-4)$, where $D = [0, 4]$

Find all values of a so that the proportion of D where $f(x) > 0$ is equal to the proportion of D where $f(x) < 0$.

Answer

$\boxed{\{1, 3\}}$

Explanation

Firstly, note that for $a \leq 0$ and $a \geq 4$ the partition of $f(x)$ on $[0, 4]$ will remain unchanged (although the ratio will have been flipped, it is still not equal to $\frac{1}{2}$.

Consider all integer values of a on $(0, 4)$

$a = 1$	$\begin{matrix} \vert & - & \vert & - & \vert & + & \vert \\ 0 & & 1 & & 2 & & 4 \end{matrix}$	$r = \frac{1}{2}$
$a = 2$	$\begin{matrix} \vert & - & \vert & + & \vert & + & \vert \\ 0 & & 1 & & 2 & & 4 \end{matrix}$	$r = \frac{3}{4}$
$a = 3$	$\begin{matrix} \vert & - & \vert & + & \vert & - & \vert & + & \vert \\ 0 & & 1 & & 2 & & 3 & & 4 \end{matrix}$	$r = \frac{1}{2}$

Further note that for $a \in (0, 2)$, $f(x)$ on $(2, 4)$ is always positive, so $f(x)$ on $(0, 2)$ must be negative (except possibly at a countable set of points) which can only occur when $a = 1$. Also for $a \in (2, 4)$, $f(x)$ on $(0, 1)$ is negative and on $(1, 2)$ is positive, thus $f(x)$ on $(2, 4)$ must be half positive and half negative which only occurs at $a = 3$.