Problem 6

Consider the function:

$f(x) = 3^{\frac{\sqrt{4-x^2}}{\sqrt{10-x^2}}}$

Let a, b be the number of integers in the domain and range respectively,

What is ab?

Answer

$\boxed{10}$

Explanation

The domain is the intersection of 3 sets:
$D^1_{f(x)} = \{x \mid 4 - x^2 \geq 0\}$
$D^2_{f(x)} = \{x \mid 10 - x^2 \geq 0\}$
$D^3_{f(x)} = \{x \mid 10 - x^2 \neq 0\}$

$D_{f(x)} = D^1_{f(x)} \cap D^2_{f(x)} \cap D^3_{f(x)} \to [-2, 2]$

Since $f(x)$ is even, the range is unaffected by restricting the domain to $[0, 2]$. Observing that $f(x)$ is now a one-to-one correspondence, we can find the range by finding the domain of of $f^{-1}(y)$ on $\{x \mid 0 \leq x \leq 2\}$

We then find that:

$f^{-1}(y) = \sqrt{\frac{2(18-5y^2)}{9-y^2}}, y \geq 0$

The domain of $f^{-1}(y)$ is the intersection of 4 sets:
$D^1_{f^{-1}(y)} = \{y \mid 18 - 5y^2 \geq 0\}$
$D^2_{f^{-1}(y)} = \{y \mid 9 - y^2 \geq 0\}$
$D^3_{f^{-1}(y)} = \{y \mid 9 - y^2 \neq 0\}$
$D^4_{f^{-1}(y)} = \{y \mid y \geq 0\}$

$R_{f(x)} = D_{f^{-1}(y)} = D^1_{f^{-1}(y)} \cap D^2_{f^{-1}(y)} \cap D^3_{f^{-1}(y)} \cap D^4_{f^{-1}(y)} \to [\ 0, 3\sqrt{\frac{2}{5}}\ ]$

Therefore $a = 5, b = 2 \to ab = 10$